

On the rigid string contribution to the interquark potential. ¹

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25 June 2014

¹M. Caselle, M. Panero, R. Pellegrini, D. Vadicchino arXiv:1406.5127

Summary:

- 1 Introduction and motivation
- 2 Rigid String corrections
- 3 The 3d $U(1)$ model
- 4 Conclusions

Universality of effective string corrections.

- The Effective String action is strongly constrained by Lorentz invariance. **The first few orders of the action are universal and coincide with those of the Nambu-Goto action.** This explains why *at large enough interquark distance* N.-G. describes so well the behaviour of Wilson loops or Polyakov Loop correlators.^{1 2 3}
- However at shorter distances deviations with respect to N.-G. cannot be neglected and may be a signature of the "true" fundamental string description of Yang Mills theories.

¹M. Luscher and P. Weisz JHEP07(2004)014

²H. B. Meyer JHEP05(2006)066

³O. Aharony and M. Field JHEP01(2011)065

Effective string action

The most general action for the effective string can be written as a low energy expansion in the number of derivatives of the transverse fields ("physical gauge").

$$S = S_{cl} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + c_2 (\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3 (\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] + S_b,$$

where:

- S_{cl} describes the usual ("classical") perimeter-area term.
- S_b is the boundary contribution characterizing the open string
- $X_i(\xi_0, \xi_1)$ ($i = 1, \dots, d - 2$) parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand
- ξ_0, ξ_1 are the world-sheet coordinates.
- In the Nambu-Goto case $c_2 = \frac{1}{8}$ and $c_3 = -\frac{1}{4}$

Effective string and spacetime symmetries.

- Symmetries of the action must hold in the low energy regime. \implies Poincaré symmetry is broken spontaneously.
- String vacuum is not Poincaré invariant.

$ISO(D - 1, 1) \rightarrow SO(D - 2) \otimes ISO(1, 1). \implies 3(D - 2)$ Goldstone bosons?

Only $D - 2$ transverse fluctuations of the string, where are the remaining Goldstone bosons?

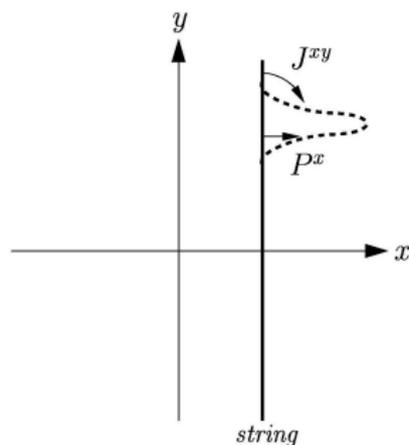
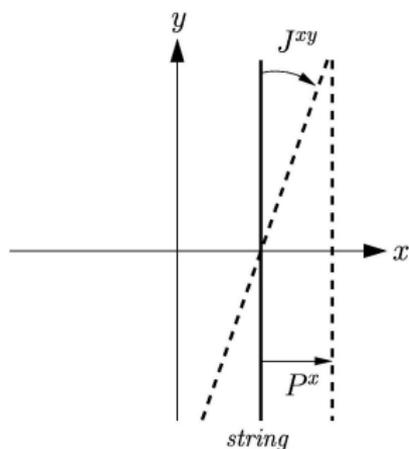
Goldstone's theorem states that there is a massless mode for each broken symmetry generator, but this counting cannot be naively extended to the case of spontaneously broken spacetime symmetries¹.

¹I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's theorem" Phys.Rev.Lett. 88 (2002) 101602

Effective string and spacetime symmetries.

The remaining $2(D - 2)$ Lorentz transformations are realized non-linearly and induce a set of recurrence relations among different terms in the action!¹

$$\delta_{\epsilon}^{bj} X_i = \epsilon (-\delta_{ij} \xi_b - X_j \partial_b X_i)$$



¹I. Low and A.V. Manohar, "Spontaneously broken spacetime symmetries and Goldstone's

Non-linear realization and long-string expansion.

A few rules to construct the most general effective string action:

- Broken **translations**:
 $X^i \rightarrow X^i + a^i$. \implies Only **field derivatives** in the effective action.
- Broken **rotation** in the plane (1, 2):

$$\delta_\epsilon^{bj} X_i = \epsilon (-\delta_{ij} \xi_b - X_j \partial_b X_i)$$

Number of derivatives minus number of fields (**weight**) preserved.

Fields and coordinates rescaling \implies **Derivative expansion**:

$$\partial_a X^i \longrightarrow \frac{1}{\sqrt{\sigma} R} \partial_a X^i.$$

Variations by broken rotation mix orders \implies **Recurrence relations**.

$ISO(1, 1)$ and $SO(D - 2)$ invariance \implies **Contraction** of indices.

Effective string action is strongly constrained! ¹ ² ³

- the terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- In three dimensions the first allowed correction to the Nambu-Goto action turns out to be an eight derivatives term which gives a contribution to the interquark potential of the order $1/R^7$
- The fact that the first deviations from the Nambu-Goto string are of high order, especially in $d = 3$, explains why in early Monte Carlo calculations a good agreement with the Nambu-Goto string was observed.
- **The effective string action is much more predictive than typical effective models in particle physics!**

¹M. Luscher and P. Weisz JHEP07(2004)014

²H. B. Meyer JHEP05(2006)066

³O. Aharony and M. Field JHEP01(2011)065

Geometrical description.

A more intuitive geometrical description of this result is obtained using the original string action, without fixing the physical gauge.

The effective action is given by the most general mapping:

$$X^\mu : \mathcal{M} \rightarrow \mathbb{R}^D, \quad \mu = 0, \dots, D-1$$

- \mathcal{M} : two-dimensional surface describing the worldsheet of the string
- \mathbb{R}^D : (flat) D dimensional target space \mathbb{R}^D of the gauge theory.

Main Result ¹ :

- The first few terms of the action compatible with Poincaré and parity invariance are suitable combinations of geometric invariants constructed from the induced metric $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$.
- These terms can be classified according to their **weight**, i.e. the difference between the number of derivatives minus the number of fields X^μ

¹O. Aharony and Z. Komargodski, JHEP **1305** (2013) 118

Geometrical description.

- The only term of weight zero is the Nambu-Goto action

$$S_{\text{NG}} = \sigma \int d^2\xi \sqrt{g} ,$$

where $g \equiv \det(g_{\alpha\beta})$.

- This term has a natural geometric interpretation: it measures the area swept out by the worldsheet in space-time.
- Fixing the physical gauge one finds (choosing an euclidean metric)

$$S = \sigma \int d^2\xi \sqrt{\det(\eta_{\alpha\beta} + \partial_\alpha X \cdot \partial_\beta X)}$$

$$\sim \sigma RT + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{8} (\partial_\alpha X \cdot \partial^\alpha X)^2 - \frac{1}{4} (\partial_\alpha X \cdot \partial_\beta X)^2 + \dots \right] ,$$

Geometrical description.

- At weight two, two new contributions appear:

$$S_{2,\mathcal{R}} = \gamma \int d^2\xi \sqrt{g} \mathcal{R},$$

$$S_{2,K} = \alpha \int d^2\xi \sqrt{g} K^2,$$

where \mathcal{R} denotes the Ricci scalar constructed from the induced metric, and $K \equiv \Delta(g)X$ is the extrinsic curvature, where $\Delta(g)$ is the Laplacian in the space with metric $g_{\alpha\beta}$.

However both these terms can be neglected!

- \mathcal{R} is topological in two dimensions and, since in the long string limit in which we are interested we do not expect topologically changing fluctuations, its contribution is constant and can be neglected.
- K^2 is proportional to the equation of motion of the Nambu-Goto Lagrangian and can be eliminated by a suitable field redefinition.

Geometrical description.

Thus the first non trivial terms appear at level four and contribute to the interquark potential with terms proportional to $1/R^7$ in agreement with the derivation in the physical gauge.

However something must be missing in the picture since high precision simulations of various 3d gauge models show large deviations with respect to the Nambu-Goto prediction, which turn out to be much stronger than the expected $1/R^7$ corrections!

- Interface free energy in the 3d Ising model¹: Torus geometry, no boundary corrections
- Excited string states of SU(N) Yang-Mills theories²
- Interquark Potential in the 3d U(1) gauge model ³

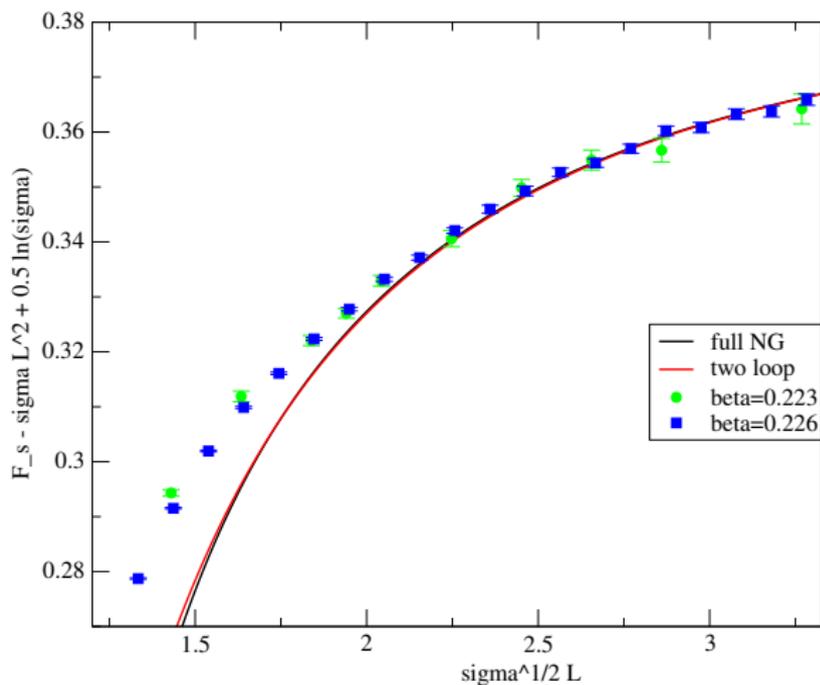
¹M. Caselle, M. Hasenbusch and M. Panero, *JHEP* **0709** (2007) 117, arXiv:0707.0055

²A. Athenodorou, B. Bringoltz and M. Teper, *JHEP* **02**, 030 (2011), arXiv:1007.4720 .

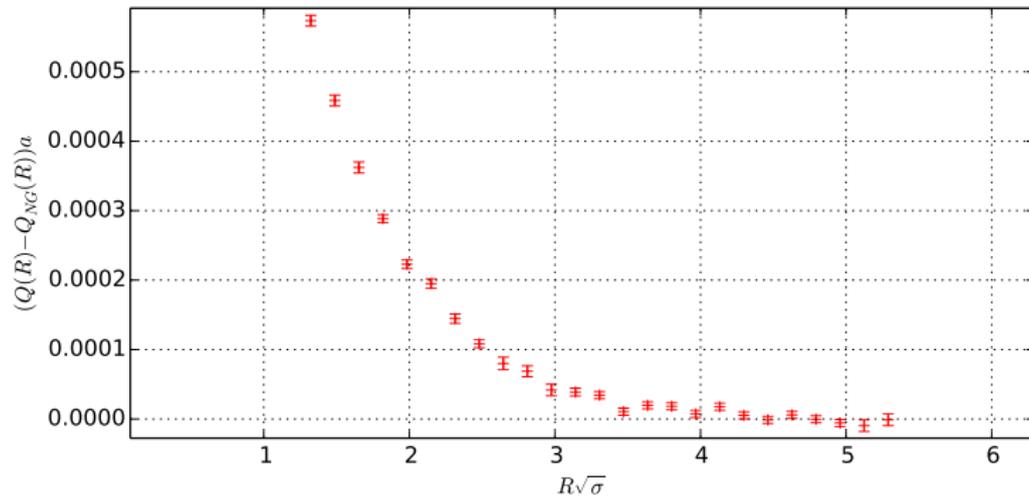
³D. Vadicchino, M. Caselle, R. Pellegrini and M. Panero, arXiv:1311.4071

3d Ising interfaces

square interfaces



3d U(1) Polyakov loops correlators



Rigid String.

Our proposal: The extrinsic curvature term vanishes at tree level but may give a non trivial contribution at one loop. ¹

Thus, the effective string action up to term proportional to $1/R^4$ is

$$S = S_{\text{NG}} + S_{2,K} + S_b$$

with:

$$S_{\text{NG}} \simeq S_{\text{cl}} + \frac{\sigma}{2} \int d^2\xi \left[\partial_\alpha X \cdot \partial^\alpha X - \frac{1}{4} (\partial_\alpha X \cdot \partial^\alpha X)^2 \right],$$

$$S_{2,K} \simeq \alpha \int d^2\xi (\Delta X)^2,$$

$$S_b \simeq b_2 \int d\xi_0 [\partial_1 \partial_0 X \cdot \partial_1 \partial_0 X].$$

Thus we are left with three free parameters (σ , α and b_2) which will be fitted comparing with the numerical data.

¹M. Caselle, M. Panero, R. Pellegrini, D. Vadicchino arXiv:1406.5127

Zeta-function regularization of the extrinsic curvature action

The Gaussian part of the action is

$$S = \sigma \int_0^{N_t} dt \int_0^R dr \left[1 + \frac{1}{2} \partial_\alpha X \cdot \partial^\alpha X \right] + \alpha \int_0^{N_t} dt \int_0^R dr (\Delta X)^2,$$

where R denotes the interquark distance, N_t is the system size in the Euclidean time direction and Δ is the two-dimensional Laplace operator

$$\Delta = \partial^2 / \partial t^2 + \partial^2 / \partial r^2.$$

The interquark potential is defined as

$$V(R) = - \lim_{N_t \rightarrow \infty} \frac{1}{N_t} \ln \left\{ \int [DX] e^{-S[X]} \right\},$$

Zeta-function regularization

The Gaussian part of the action can be rewritten as

$$S = \sigma \int_0^{N_t} dt \int_0^R dr \left[1 + \frac{1}{2} X \left(1 - \frac{2\alpha}{\sigma} \Delta \right) (-\Delta) X \right].$$

Carrying out the Gaussian integration, one obtains

$$V(R) = \lim_{N_t \rightarrow \infty} \left\{ \sigma R + \frac{1}{2N_t} \text{Tr} \ln(-\Delta) + \frac{1}{2N_t} \text{Tr} \ln \left(1 - \frac{1}{m^2} \Delta \right) \right\}.$$

The parameter $m = \frac{\sigma}{2\alpha}$, which has the dimensions of a mass, encodes the contribution due to the extrinsic curvature.

Zeta-function regularization

The operator traces are singular but can be evaluated using the zeta-function regularization:

$$V(R) = \sigma R + V_{\text{NG}}(R) + V_{\text{ext}}(R, m),$$

where $V_{\text{NG}}(R)$ and $V_{\text{ext}}(R, m)$ are the Gaussian limit of the Nambu-Goto and of the extrinsic curvature contributions respectively:

$$V_{\text{NG}}(R) \equiv \lim_{N_t \rightarrow \infty} \frac{1}{2N_t} \text{Tr} \ln(-\Delta) = -\frac{\pi}{24R},$$

$$V_{\text{ext}}(R, m) \equiv \lim_{N_t \rightarrow \infty} \frac{1}{2N_t} \text{Tr} \ln \left(1 - \frac{1}{m} \Delta \right) = -\frac{m}{2\pi} \sum_{n=1}^{\infty} \frac{K_1(2nmR)}{n},$$

where $K_\alpha(z)$ denotes a modified Bessel function of the second kind.

Analytical properties of $V_{\text{ext}}(R, m)$

$V_{\text{ext}}(R, m)$ has a logarithmic branching point at $R = 0$ and a set of square-root singularities for negative values of $(mR)^2$. The first is located at $(mR)^2 = -\pi^2$, and defines the radius of convergence of the low mR expansion

$$V_{\text{ext}}(R, m) = -\frac{\pi}{24R} + \frac{m}{4} + \frac{m^2 R}{4\pi} \left(\ln \frac{mR}{2\pi} + \gamma_E - \frac{1}{2} \right) + \frac{m^2 R}{2\pi} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right) \zeta(2n+1)}{\Gamma(n+2)\Gamma\left(n-\frac{1}{2}\right)} \left(\frac{mR}{\pi}\right)^{2n},$$

where $\gamma_E = 0.5772156649\dots$ is the Euler-Mascheroni constant and $\zeta(x)$ denotes the Riemann zeta function.

In the large- R limit $V_{\text{ext}}(R, m)$ decreases exponentially. Its behavior is dominated by the lowest-index Bessel function appearing in the sum:

$$V_{\text{ext}}(R, m) \simeq -\sqrt{\frac{m}{16\pi R}} e^{-2mR} \quad \text{for } R \gg \frac{1}{m}.$$

This is the typical behavior expected for a massive perturbation of a 2d CFT,

Analytical properties of $V_{\text{ext}}(R, m)$

- $V_{\text{ext}}(R, m)$ can be understood as a **massive perturbation of the $c = 1$ free bosonic theory**. In fact, the combination

$$c_0(mR) = -\frac{24R}{\pi} V_{\text{ext}}(R, m)$$

coincides with the ground state scaling function $c_0(mR)$ describing this perturbation.

- $c_0(mR)$ is a monotonically decreasing function of its argument and interpolates between 1 for $mR = 0$ and 0 for $mR \rightarrow \infty$.
- Notice the analogy with the Nambu-Goto case: while the Nambu-Goto model can be described as an irrelevant *massless* perturbation of the $c = 1$ free bosonic 2d CFT^{1, 2}, the rigid string is described by a relevant *massive* perturbation of the same CFT.
- In the $mR \rightarrow 0$ limit, the free bosonic theory is recovered: thus we find a second “Lüscher” term, in addition to the one from $V_{\text{NG}}(R)$.

¹S.Dubovsky, R. Flauger, V. Gorbenko JHEP1209 (2012) 044

²M. Caselle, D. Fioravanti, F. Gliozzi, R. Tateo JHEP07 (2013) 071

Main differences between the NG and rigid strings

- The field density profile around the string is (almost) a Gaussian in the case of a Nambu-Goto string, while it decreases exponentially in the rigid string case. This exponential defines a new scale, known as the London penetration length in condensed matter theory, and as intrinsic width in confining gauge theories
- While in the Nambu-Goto case the string width increases logarithmically with the interquark distance at zero temperature and linearly at high temperature, the intrinsic width of the rigid string is constant
- At very short distances the coefficient of the Lüscher term is doubled.

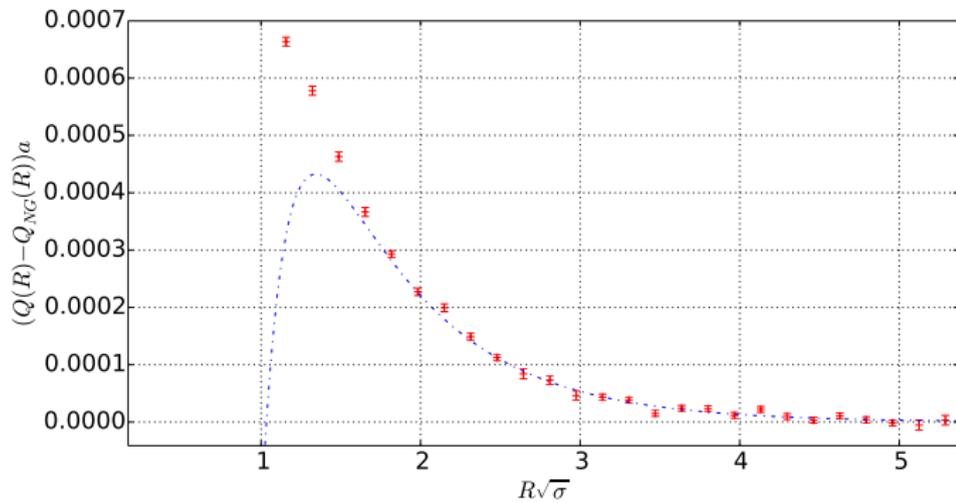
3d U(1) model

- Using duality Polyakov ¹ was able to give a heuristic string description of the model with a string action combining both the Nambu-Goto and the extrinsic curvature terms. Following his derivation one finds that m is proportional to the "glueball" mass of the U(1) model
- Including in the fits also the rigidity term dramatically improves the quality of the fits, moreover the rigidity parameter scales exactly as predicted by Polyakov. Details on this analysis in the next talk by Davide Vadicchino
- The relevance of the rigid string correction for the U(1) theory is mainly due to the nontrivial scaling behaviour of $m_0/\sqrt{\sigma}$ in this case.

$$\frac{m_0}{\sqrt{\sigma}} = \frac{2\pi c_0}{\sqrt{c_\sigma}} (2\pi\beta)^{3/4} e^{-\pi^2 v(0)\beta/2},$$

As β increases toward the continuum limit the rigidity term becomes more and more important and **dominates in the continuum limit!**

¹Polyakov, Nucl. Phys. B486 (1997) 23



Conclusions

- The Effective String action is strongly constrained by Lorentz invariance. The first few orders of the bulk and of the boundary action are universal
- However deviations from the expected Nambu-Goto behaviour (in particular in the 3d $U(1)$ model suggest that something is missing in the picture
- In the case of the 3d $U(1)$ model, including also a rigidity term allows to perfectly fit the data.
- While the NG action was shown to be described by a **massless perturbation of the $c = 1$ free field theory** (perturbed by the irrelevant operator $T\bar{T}$), the rigid string correction can be described as a **massive perturbation of the $c = 1$ free field theory**.
- The 3d $U(1)$ lattice model turns out to be a perfect laboratory to study the cross-over from a purely Nambu-Goto string at low β to a purely rigid string at large β .

Conclusions

- The relevance of the rigid string correction for the $U(1)$ theory is mainly due to the nontrivial scaling behaviour of $m_0/\sqrt{\sigma}$.
- In $SU(N)$ or Ising LGTs $m_0/\sqrt{\sigma}$ is constant, however it is well possible that a rigidity term is present also in these theories.
- It would not be dominant in the continuum limit but it could explain the short scale deviations observed in recent simulations and the numerically observed London penetration term in the string width, which otherwise would be incompatible with a Nambu-Goto string.